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On the Grassmann modules for the symplectic groups

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ABSTRACT

Let V be a $2n$-dimensional vector space ($n \geqslant 1$) over a field $\mathbb{K}$ equipped with a nondegenerate alternating bilinear form f, and let $G \cong Sp(2n, \mathbb{K})$ denote the group of isometries of (V, f). For every $k \in \{1, \ldots, n\}$, there exists a natural representation of G on the subspace W_k of $\bigwedge^k V$ generated by all vectors $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k$ such that $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ is totally isotropic with respect to f. With the aid of linear algebra, we prove some properties of this representation. In particular, we determine a necessary and sufficient condition for the representation to be irreducible and characterize the largest proper G-submodule. These facts allow us to determine when the Grassmann embedding of the symplectic dual polar space $DW(2n-1, \mathbb{K})$ is isomorphic to its minimal full polarized embedding.

1. Notation and main results

Let $n \geqslant 1$ and let $\mathbb{K}$ be a field. Put $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$. The characteristic of $\mathbb{K}$ is either 0 or a prime number p. If $\mathrm{char}(\mathbb{K}) = p$, then for every strictly positive integer m let h_m be the largest nonnegative integer such that $p^{h_m} \mid m$, and for every $\epsilon \in \mathbb{N}$ let $N_{\epsilon,p}$ denote the smallest multiple of $p^{1+h_{\epsilon+1}}$ greater than ϵ (and hence also greater than $\epsilon + 1$).

Let V be a $2n$-dimensional vector space over $\mathbb{K}$ equipped with a nondegenerate alternating bilinear form f. An ordered basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V is called a *hyperbolic basis* of V if $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$. Let G denote the group of all isometries of (V, f), i.e. the set of all $\theta \in GL(V)$ satisfying $f(\theta(\bar{x}), \theta(\bar{y})) = f(\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in V$. Then $G \cong Sp(2n, \mathbb{K})$. The elements of G are precisely the elements of $GL(V)$ which map hyperbolic bases of V to hyperbolic bases of V.

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For every $k \in \{0, \ldots, 2n\}$, let $\bigwedge^k V$ be the k-th exterior power of V. Then $\bigwedge^0 V = \mathbb{K}$ and $\bigwedge^1 V = V$. If $k \in \{1, \ldots, 2n\}$, then for every $\theta \in GL(V)$, there exists a unique $\tilde{\theta}_k \in GL(\bigwedge^k V)$ such that $\tilde{\theta}_k(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \cdots \wedge \theta(\bar{v}_k)$ for all vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$.

Now, let $k \in \{1, \ldots, n\}$ and let W_k denote the subspace of $\bigwedge^k V$ generated by all vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ such that $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ is a k-dimensional subspace of V which is totally isotropic with respect to f. The dimension of W_k is equal to $\binom{2n}{k} - \binom{2n}{k-2}$. For every $\theta \in G$, $\tilde{\theta}_k$ stabilizes W_k and hence the map $\theta \mapsto \tilde{\theta}_{k|W_k}$ defines a representation $\mathcal{R}_k$ of the group $G \cong Sp(2n, \mathbb{K})$ on the $(\binom{2n}{k} - \binom{2n}{k-2})$-dimensional vector space W_k. We call the corresponding $\mathbb{K}G$-module a *Grassmann module for* $Sp(2n, \mathbb{K})$. Put $\widetilde{G}_k := \{\tilde{\theta}_{k|W_k} \mid \theta \in G\}$.

Let R_k denote the set of all vectors $\alpha \in W_k$ such that $\alpha \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n = 0$ for any n linearly independent vectors $\bar{v}_1, \ldots, \bar{v}_n$ of V such that $\langle \bar{v}_1, \ldots, \bar{v}_n \rangle$ is totally isotropic with respect to f. The subspace R_k is stabilized by any element of $\widetilde{G}_k$. Hence, the representation $\mathcal{R}_k$ induces a representation $\mathcal{R}_k^{(1)}$ of G on the quotient space W_k/R_k and a representation $\mathcal{R}_k^{(2)}$ of G on the subspace R_k.

From the theory of Lie algebras, it follows that W_k, $k \in \{1, \ldots, n\}$, has a largest proper G-submodule. In the case that $\mathbb{K}$ is a field of odd characteristic, a (complicated) recursive formula for the dimension of the largest proper G-submodule has been given in [6, Theorem 2(i)]. This result was generalized to the case $\mathrm{char}(\mathbb{K}) = 2$ by Adamovich in her PhD thesis, see [1] and [2]. With the aid of linear algebra, we will prove the following in Section 3.7:

Theorem 1.1. *If U is a proper subspace of W_k stabilized by $\widetilde{G}_k$, then $U \subseteq R_k$.*

The following is an immediate corollary of Theorem 1.1.

Corollary 1.2.

(1) *R_k is the largest proper G-submodule of W_k.*
(2) *The representation $\mathcal{R}_k^{(1)}$ is irreducible.*
(3) *The representation $\mathcal{R}_k$ is irreducible if and only if $R_k = 0$.*

Remark. In the special case that $k = n$, the conclusion mentioned in Theorem 1.1 was also obtained in [3] as part of a more general result regarding full polarized embeddings of dual polar spaces.

If $\mathit{char}(\mathbb{K}) = p$ is odd, then Premet and Suprunenko [6, Theorem 2(iv)] gave necessarily and sufficient conditions for the representation $\mathcal{R}_k$ to be irreducible. They proved the following.

Proposition 1.3. *(See [6].) If $\mathit{char}(\mathbb{K}) = p$ is odd, then the representation $\mathcal{R}_k$ is irreducible if and only if p does not divide the number*

$$\prod_j \binom{n - \frac{j+k}{2} + 1}{n - k + 1},$$

where the product ranges over all $j \in \{0, \ldots, k\}$ having the same parity as k.

The following is a corollary of Proposition 1.3.

Corollary 1.4. *Suppose $\mathit{char}(\mathbb{K}) = p$ is odd. Let $\epsilon \in \mathbb{N}$ be fixed and $n > \epsilon$ be variable. Then there exists an integer $n^* > \epsilon$ such that $\mathcal{R}_{n-\epsilon}$ is irreducible if and only if $\epsilon < n < n^*$. Moreover, $n^* = 2(N_{\epsilon,p} - 1) - \epsilon$.*

Proof. We will make use of Proposition 1.3. Notice that if $k = n - \epsilon$, then

$$\prod_j \binom{n-\frac{j+k}{2}+1}{n-k+1} = \prod_j \binom{\epsilon+1+\frac{k-j}{2}}{\epsilon+1},$$

where the product ranges over all $j \in \{0,\dots,k\}$ having the same parity as k. So, there exists a number n^* as in the statement of the corollary if and only if there exists a natural number m such that $g(m) := \binom{\epsilon+1+m}{\epsilon+1}$ is divisible by p. Now, $g(0)=1$, $h_{g(0)}=0$ and $\frac{g(m)}{g(m-1)} = \frac{\epsilon+1+m}{m}$ for every $m \in \mathbb{N}\setminus\{0\}$. We have $h_{g(m)} > h_{g(m-1)}$ if and only if $h_{\epsilon+1+m} > h_m$, or equivalently, if and only if $h_{\epsilon+1+m} > h_{\epsilon+1}$. Hence, the smallest value of m for which $h_{g(m)} > 0$ is equal to $N_{\epsilon,p} - \epsilon - 1$. So, the values of $k^* := n^* - \epsilon$ and n^* are respectively equal to $2(N_{\epsilon,p} - \epsilon - 1)$ and $n^* = k^* + \epsilon = 2(N_{\epsilon,p} - 1) - \epsilon$. □

The proof of Proposition 1.3 given in [6] relies very much on the theory of Lie algebras and the representation theory for the symmetric groups (Specht modules). Using only linear algebra, we give a proof of the following facts in Section 3.8:

Theorem 1.5.

(1) *If char*$(\mathbb{K})=0$, *then* $R_k = 0$ *for every* $k \in \{1,\dots,n\}$.
(2) *Suppose char*$(\mathbb{K})=p$ *and* $\epsilon \in \mathbb{N}$. *Then* (*for fixed* ϵ *and variable* $n > \epsilon$) $R_{n-\epsilon} = 0$ *if and only if* $\epsilon < n < n^* := 2(N_{\epsilon,p}-1)-\epsilon$. *If* $n = n^*$, *then* $\dim(R_{n-\epsilon}) = 1$. *If* $n = n^*+1$, *then* $\dim(R_{n-\epsilon}) = 2n$. *For all* $n > \epsilon$, $\dim(R_{n+1-\epsilon}) \geqslant 2\cdot\dim(R_{n-\epsilon})$.
(3) *Suppose char*$(\mathbb{K})=p$ *and* $\epsilon \in \mathbb{N}$. *If* $n = n^*+1$, *then the representation* $\mathcal{R}^{(2)}_{n-\epsilon}$ *is isomorphic to the natural representation of* $G \cong Sp(2n,\mathbb{K})$ *on* V.

By Corollary 1.2(3) and Theorem 1.5, we have:

Corollary 1.6. *The representation* $\mathcal{R}_k$ *is irreducible if and only if either char*$(\mathbb{K})=0$ *or* (*char*$(\mathbb{K})=p$ *and* $n < 2(N_{\epsilon,p}-1)-\epsilon$), *where* $\epsilon = n-k$.

2. Application to projective embeddings of symplectic dual polar spaces

Let Π be a polar space (Tits [7]; Veldkamp [8]) of rank $n \geqslant 2$. For every singular subspace ω of Π, let C_ω denote the set of all maximal (i.e. $(n-1)$-dimensional) singular subspaces of Π containing ω. With Π, there is associated a point-line geometry $\Delta = (\mathcal{P},\mathcal{L})$ which is called a *dual polar space*. The point-set $\mathcal{P}$ of Δ consists of all maximal singular subspaces of Π, and the line-set $\mathcal{L}$ consists of all sets C_ω, where ω is some $(n-2)$-dimensional singular subspace of Π. If ω_1 and ω_2 are two points of Δ, then $\mathrm{d}(\omega_1,\omega_2)$ denotes the distance between ω_1 and ω_2 in the collinearity graph of Δ. The distance $\mathrm{d}(\omega_1,\omega_2)$ is equal to $n-1-\dim(\omega_1\cap\omega_2)$. So, the collinearity graph of Δ has diameter n. For every point ω of Δ and every $i \in \mathbb{N}$, $\Delta_i(\omega)$ denotes the set of points of Δ at distance i from ω.

A *full* (*projective*) *embedding* of a point-line geometry $\mathcal{S}$ is an injective mapping e from the point-set $\mathcal{P}$ of $\mathcal{S}$ to the point-set of a projective space Σ satisfying (i) $\langle e(\mathcal{P})\rangle_\Sigma = \Sigma$ and (ii) $e(L)$ is a line of Σ for every line L of $\mathcal{S}$. Two full embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* if there exists an isomorphism $\kappa : \Sigma_1 \to \Sigma_2$ such that $e_2 = \kappa \circ e_1$.

A full embedding $e : \Delta \to \Sigma$ of a thick dual polar space $\Delta = (\mathcal{P},\mathcal{L})$ is called *polarized* if $\langle e(\mathcal{P}\setminus\Delta_n(x))\rangle_\Sigma$ is a hyperplane of Σ for every point x of Δ. The intersection $\mathcal{N}_e$ of all subspaces $\langle e(\mathcal{P}\setminus\Delta_n(x))\rangle_\Sigma$, $x \in \mathcal{P}$, is called the *nucleus* of the polarized embedding e. The mapping $x \mapsto \langle \mathcal{N}_e, e(x)\rangle_\Sigma$ defines a full polarized embedding $e/\mathcal{N}_e$ of Δ into the quotient space $\Sigma/\mathcal{N}_e$. If e_1 and e_2 are two full polarized embeddings of Δ, then $e_1/\mathcal{N}_{e_1}$ and $e_2/\mathcal{N}_{e_2}$ are isomorphic, see Cardinali, De Bruyn and Pasini [4]. The embedding $e/\mathcal{N}_e$, where e is an arbitrary full polarized embedding of Δ, is called the *minimal full polarized embedding of* Δ.

As in Section 1, let V be a $2n$-dimensional vector space over a field $\mathbb{K}$ equipped with a nondegenerate alternating bilinear form f. We suppose that $n \geqslant 2$. With the pair (V,f), there is associated a symplectic polar space $W(2n-1,\mathbb{K})$ and a symplectic dual polar space $DW(2n-1,\mathbb{K})$. The singular subspaces of $W(2n-1,\mathbb{K})$ are the subspaces of $PG(2n-1,\mathbb{K})$ which are totally isotropic with respect to the symplectic polarity of $PG(V)$ defined by f. Now, for every point $\omega = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ of $DW(2n-1,\mathbb{K})$, let $e_{gr}(\omega)$ denote the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \rangle$ of $PG(\bigwedge^n V)$. Then e_{gr} defines a full polarized embedding of $DW(2n-1,\mathbb{K})$ into $PG(W_n)$, where W_n is the subspace of $\bigwedge^n V$ as defined in Section 1. This embedding is called the *Grassmann embedding* of $DW(2n-1,\mathbb{K})$. The nucleus $\mathcal{N}_{e_{gr}}$ of e_{gr} is equal to $PG(R_n)$, where R_n is the subspace of W_n as defined in Section 1. The following is a corollary of Theorem 1.5 (take $\epsilon = 0$).

Corollary 2.1.

(1) *If* $\mathrm{char}(\mathbb{K}) = 0$, *then the nucleus of the Grassmann embedding of* $DW(2n-1,\mathbb{K})$ *is empty.*
(2) *If* $\mathrm{char}(\mathbb{K}) = p$, *then the nucleus of the Grassmann embedding of* $DW(2n-1,\mathbb{K})$ *is empty if and only if* $n < 2(p-1)$. *If* $n = 2(p-1)$, *then the nucleus is a point. If* $n = 2p-1$, *then the nucleus is a subspace of dimension* $2n-1$.
(3) *The Grassmann embedding of* $DW(2n-1,\mathbb{K})$ *is isomorphic to the minimal full polarized embedding of* $DW(2n-1,\mathbb{K})$ *if and only if either* $\mathrm{char}(\mathbb{K}) = 0$ *or* ($\mathrm{char}(\mathbb{K}) = p$ *and* $n < 2(p-1)$).

3. Proofs of Theorems 1.1 and 1.5

In this section, we will continue with the notation introduced in Section 1. Recall that V is a $2n$-dimensional vector space ($n \geqslant 1$) over $\mathbb{K}$ which is equipped with a nondegenerate alternating bilinear form f.

3.1. Hyperbolic bases of V

If $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V, then:

(1) for every permutation σ of $\{1,\ldots,n\}$, also $(\bar{e}_{\sigma(1)}, \bar{f}_{\sigma(1)}, \ldots, \bar{e}_{\sigma(n)}, \bar{f}_{\sigma(n)})$ is a hyperbolic basis of V;
(2) for every $\lambda \in \mathbb{K}^*$, also $(\frac{\bar{e}_1}{\lambda}, \lambda\bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V;
(3) for every $\lambda \in \mathbb{K}$, also $(\bar{e}_1 + \lambda\bar{e}_2, \bar{f}_1, \bar{e}_2, -\lambda\bar{f}_1 + \bar{f}_2, \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V;
(4) for every $\lambda \in \mathbb{K}$, also $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n, \bar{f}_n + \lambda\bar{e}_n)$ is a hyperbolic basis of V;
(5) for every $\lambda \in \mathbb{K}$, also $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n + \lambda\bar{f}_n, \bar{f}_n)$ is a hyperbolic basis of V.

For every $i \in \{1,2,3,4,5\}$, let Ω_i denote the set of all ordered pairs (B_1, B_2) of hyperbolic bases of V such that B_2 can be obtained from B_1 as described in (i) above. The following lemma was proved in De Bruyn [5, Lemma 2.1].

Lemma 3.1. *(See [5].) If B and B' are two hyperbolic bases of V, then there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ ($k \geqslant 0$) of V such that $B_0 = B$, $B_k = B'$ and $(B_{i-1}, B_i) \in \Omega_1 \cup \cdots \cup \Omega_5$ for every $i \in \{1,\ldots,k\}$.*

Lemma 3.2. *Let $k, l \in \{0,\ldots,2n\}$ with $1 \leqslant k \leqslant 2n$ and $0 \leqslant l \leqslant \min(n,k)$. Let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ be k linearly independent vectors of V such that $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ is totally isotropic with respect to f. Then there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V, a $\lambda \in \mathbb{K}^*$, and an $m \in \{\max(0,k-n), \ldots, \min(k-l, \lfloor \frac{k}{2} \rfloor)\}$ such that*

$$\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}.$$

Proof. Let R denote the radical of the alternating bilinear form of $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ induced by f and let Z be a subspace of $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ complementary to R. The alternating bilinear form f_Z of Z induced by f is nondegenerate. Hence, $\dim(Z)$ is even, say $\dim(Z) = 2m \geqslant 0$. Let $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_m, \bar{f}_m)$ be a hyperbolic basis of Z with respect to f_Z. Consider a basis $\{\bar{e}_{m+1}, \ldots, \bar{e}_{k-m}\}$ of R. Then $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_m, \bar{f}_m, \bar{e}_{m+1}, \ldots, \bar{e}_{k-m})$ can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V.

Obviously, $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}$ for some $\lambda \in \mathbb{K}^*$. Clearly, $\dim(Z) = 2m \leqslant k$, i.e. $m \leqslant \lfloor \frac{k}{2} \rfloor$. Also, since the dimension of a maximal totally isotropic subspace of $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle = \langle R, Z \rangle$ is equal to $k - m$, we have $l \leqslant k - m \leqslant n$, i.e. $k - n \leqslant m \leqslant k - l$. $\square$

3.2. The linear maps $\theta_{k,l}$

For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V, for every $k \in \{1, \ldots, 2n\}$ and every $l \in \{0, \ldots, \lfloor \frac{k}{2} \rfloor\}$, we now define a linear map $\theta_{k,l,B} : \bigwedge^k V \to \bigwedge^{k-2l} V$. If $m \in \mathbb{N}$ such that $\max(k-n, 0) \leqslant m \leqslant \lfloor \frac{k}{2} \rfloor$, if σ is a permutation of $\{1, \ldots, n\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(k-m)$, and if $\bar{g}_{\sigma(i)} \in \{\bar{e}_{\sigma(i)}, \bar{f}_{\sigma(i)}\}$ for every $i \in \{m+1, \ldots, k-m\}$, then we put

$$\theta_{k,l,B}\big[(\bar{e}_{\sigma(1)} \wedge \bar{f}_{\sigma(1)}) \wedge \cdots \wedge (\bar{e}_{\sigma(m)} \wedge \bar{f}_{\sigma(m)}) \wedge \bar{g}_{\sigma(m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(k-m)}\big]$$

equal to 0 if $m < l$ and equal to

$$\sum (\bar{e}_{\sigma(i_1)} \wedge \bar{f}_{\sigma(i_1)}) \wedge \cdots \wedge (\bar{e}_{\sigma(i_{m-l})} \wedge \bar{f}_{\sigma(i_{m-l})}) \wedge \bar{g}_{\sigma(m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(k-m)}$$

if $m \geqslant l$. Here, the summation ranges over all subsets $\{i_1, i_2, \ldots, i_{m-l}\}$ of size $m - l$ of $\{1, \ldots, m\}$ satisfying $i_1 < i_2 < \cdots < i_{m-l}$. Since this defines $\theta_{k,l,B}$ for every element of a basis of $\bigwedge^k V$, we have defined $\theta_{k,l,B}$ for all vectors of $\bigwedge^k V$. It is straightforward (but perhaps tedious) to verify that if B_1 and B_2 are two hyperbolic bases of V such that $(B_1, B_2) \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_5$, then $\theta_{k,l,B_1} = \theta_{k,l,B_2}$. Hence by Lemma 3.1, there exists a linear map $\theta_{k,l} : \bigwedge^k V \to \bigwedge^{k-2l} V$ such that $\theta_{k,l,B} = \theta_{k,l}$ for every hyperbolic basis B of V.

3.3. The subspaces $W_{k,l}$ of $\bigwedge^k V$

For every $k \in \{1, \ldots, 2n\}$ and every $l \in \{0, \ldots, \min(n,k)\}$, let $W_{k,l}$ be the subspace of $\bigwedge^k V$ generated by all vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$, where $\bar{v}_1, \ldots, \bar{v}_k$ are k linearly independent vectors of V such that $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ is totally isotropic with respect to f. Here, we use the following convention: if $l = 0$, then $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle = \langle - \rangle = \{\bar{o}\}$. By definition, we put $W_{0,0}$ equal to $\bigwedge^0 V = \mathbb{K}$. The following is the main result of De Bruyn [5].

Proposition 3.3. *(See [5].) For every $k \in \{0, \ldots, 2n\}$ and $l \in \{0, \ldots, \min(n,k)\}$, $\dim(W_{k,l}) = \binom{2n}{k} - \binom{2n}{2l-k-2}$.*

In the previous proposition, we used the convention that $\binom{a}{b} = 0$ for every $a \in \mathbb{N}$ and every $b \in \mathbb{Z} \setminus \{0, \ldots, a\}$. The following proposition is precisely Lemma 2.5 of [5].

Proposition 3.4. *If $k \in \{0, \ldots, 2n\}$ and $0 \leqslant l \leqslant \lceil \frac{k}{2} \rceil$, then $W_{k,l} = \bigwedge^k V$.*

More generally, we can say the following.

Theorem 3.5. *For every $k \in \{0, \ldots, 2n\}$ and every $l \in \{0, \ldots, \min(n,k)\}$, $W_{k,l} = \bigcap_{i=k-l+1}^{\lfloor \frac{k}{2} \rfloor} \ker(\theta_{k,i})$. (Here, we take the convention that if $k - l + 1 > \lfloor \frac{k}{2} \rfloor$, then the intersection equals $\bigwedge^k V$.)*

Proof. If $k - l + 1 > \lfloor \frac{k}{2} \rfloor$, or equivalently if $l \leqslant \lceil \frac{k}{2} \rceil$, then $\bigcap_{i=k-l+1}^{\lfloor \frac{k}{2} \rfloor} \ker(\theta_{k,i}) = \bigwedge^k V = W_{k,l}$ by Proposition 3.4. So, in the sequel we will suppose that $l > \lceil \frac{k}{2} \rceil$.

We first prove that $W_{k,l} \subseteq \ker(\theta_{k,i})$ for every $i \in \{k - l + 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$. It suffices to prove that $\theta_{k,i}(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k) = 0$, where $\bar{v}_1, \ldots, \bar{v}_k$ are k linearly independent vectors of V such that

$\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ is totally isotropic with respect to f. By Lemma 3.2, there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V, a $\lambda \in \mathbb{K}^*$ and an $m \in \{\max(0, k-n), \ldots, \min(k-l, \lfloor \frac{k}{2} \rfloor)\}$ such that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}$. Since $m \leqslant k-l < k-l+1 \leqslant i$, $\theta_{k,i}(\bar{v}_1 \wedge \cdots \wedge \bar{v}_k) = 0$.

By the previous paragraph, we know that $W_{k,l} \subseteq \bigcap_{i=k-l+1}^{\lfloor \frac{k}{2} \rfloor} \ker(\theta_{k,i})$. Suppose now that there exists a vector $\alpha \in \bigcap_{i=k-l+1}^{\lfloor \frac{k}{2} \rfloor} \ker(\theta_{k,i})$ which is not contained in $W_{k,l}$. Since $W_{k,l} \subseteq W_{k,l-1} \subseteq \cdots \subseteq W_{k,\lceil \frac{k}{2} \rceil} = \bigwedge^k V$, there exists an $l^* \in \{l-1, l-2, \ldots, \lceil \frac{k}{2} \rceil\}$ such that $\alpha \in W_{k,l^*}$ and $\alpha \notin W_{k,l^*+1}$. Now, let ϕ be the restriction of $\theta_{k,k-l^*}$ to the subspace W_{k,l^*} of $\bigwedge^k V$.

(1) We prove that $\phi(W_{k,l^*}) \subseteq W_{2l^*-k,2l^*-k}$. It suffices to prove that $\phi(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k) \in W_{2l^*-k,2l^*-k}$ where $\bar{v}_1, \ldots, \bar{v}_k$ are k linearly independent vectors of V such that $\langle \bar{v}_1, \ldots, \bar{v}_{l^*} \rangle$ is totally isotropic with respect to f. By Lemma 3.2, there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V, a $\lambda \in \mathbb{K}^*$ and an $m \in \{\max(0, k-n), \ldots, \min(k-l^*, \lfloor \frac{k}{2} \rfloor)\}$ such that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}$. If $m < k-l^*$, then $\phi(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k) = 0$. If $m = k-l^*$, then $\phi(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k) = \lambda \cdot \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m} \in W_{2l^*-k,2l^*-k}$.

(2) We prove that $\phi(W_{k,l^*}) = W_{2l^*-k,2l^*-k}$. It suffices to prove that $\bar{e}_1 \wedge \cdots \wedge \bar{e}_{2l^*-k} \in Im(\phi)$ for every $2l^*-k$ linearly independent vectors $\bar{e}_1, \ldots, \bar{e}_{2l^*-k}$ of V such that $\langle \bar{e}_1, \ldots, \bar{e}_{2l^*-k} \rangle$ is totally isotropic with respect to f. Now, extend $(\bar{e}_1, \ldots, \bar{e}_{2l^*-k})$ to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Put $\alpha := (\bar{e}_{2l^*-k+1} \wedge \bar{f}_{2l^*-k+1}) \wedge \cdots \wedge (\bar{e}_{l^*} \wedge \bar{f}_{l^*}) \wedge \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_{2l^*-k}$. Then $\alpha \in W_{k,l^*}$ and $\phi(\alpha) = \bar{e}_1 \wedge \cdots \wedge \bar{e}_{2l^*-k}$.

(3) We prove that $\phi(W_{k,l^*+1}) = 0$. It suffices to prove that $\phi(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k) = 0$ where $\bar{v}_1, \ldots, \bar{v}_k$ are k linearly independent vectors of V such that $\langle \bar{v}_1, \ldots, \bar{v}_{l^*+1} \rangle$ is totally isotropic with respect to f. By Lemma 3.2, there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V, a $\lambda \in \mathbb{K}^*$ and an $m \in \{\max(0, k-n), \ldots, \min(k-l^*-1, \lfloor \frac{k}{2} \rfloor)\}$ such that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}$. Since $m < k-l^*$, $\phi(\bar{v}_1 \wedge \cdots \wedge \bar{v}_k) = 0$.

(4) We prove that $\ker(\phi) = W_{k,l^*+1}$. We have $\dim(\ker(\phi)) = \dim(W_{k,l^*}) - \dim(Im(\phi)) = \dim(W_{k,l^*}) - \dim(W_{2l^*-k,2l^*-k}) = \binom{2n}{k} - \binom{2n}{2l^*-k-2} - \binom{2n}{2l^*-k} + \binom{2n}{2l^*-k-2} = \binom{2n}{k} - \binom{2n}{2l^*-k} = \dim(W_{k,l^*+1})$. Since $W_{k,l^*+1} \subseteq \ker(\phi)$, we necessarily have $\ker(\phi) = W_{k,l^*+1}$.

We are now ready to derive a contradiction. Since $\alpha \in \bigcap_{i=k-l+1}^{\lfloor \frac{k}{2} \rfloor} \ker(\theta_{k,i})$, we necessarily have $\alpha \in \ker(\phi)$. On the other hand, since $\alpha \in W_{k,l^*} \setminus W_{k,l^*+1}$, we necessarily have $\alpha \notin \ker(\phi)$ by (4) above. $\square$

3.4. An invariant vector of $\bigwedge^{2k} V$, $k \in \{0, \ldots, n\}$

We can use Lemma 3.1 to prove the existence of some invariant vector of $\bigwedge^{2k} V$, $k \in \{0, \ldots, n\}$. For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V and every $k \in \{0, \ldots, n\}$, we define

$$\alpha_{2k}(B) := \sum (\bar{e}_{i_1} \wedge \bar{f}_{i_1}) \wedge (\bar{e}_{i_2} \wedge \bar{f}_{i_2}) \wedge \cdots \wedge (\bar{e}_{i_k} \wedge \bar{f}_{i_k}),$$

where the summation ranges over all $\binom{n}{k}$ subsets $\{i_1, i_2, \ldots, i_k\}$ of size k of $\{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_k$. (By convention, $\alpha_0(B) = 1 \in \mathbb{K}$.) One can readily verify that if B_1 and B_2 are two hyperbolic bases of V such that $(B_1, B_2) \in \Omega_1 \cup \cdots \cup \Omega_5$, then $\alpha_{2k}(B_1) = \alpha_{2k}(B_2)$ for every $k \in \{0, \ldots, n\}$. Hence by Lemma 3.1, there exists a vector $\alpha^*_{2k} \in \bigwedge^{2k} V$ such that $\alpha^*_{2k} = \alpha_{2k}(B)$ for every hyperbolic basis B of V.

Lemma 3.6. *Let p be a prime and $\epsilon \in \mathbb{N}$. Then the smallest $k \in \mathbb{N} \setminus \{0\}$ for which $p \mid \binom{\epsilon+k+i}{i}$, $\forall i \in \{1, \ldots, k\}$, is equal to $N_{\epsilon,p} - \epsilon - 1$.*

Proof. (1) Suppose $k = N_{\epsilon,p} - \epsilon - 1$ and let i be an arbitrary element of $\{1, \ldots, k\}$. Then

$$\binom{\epsilon+k+i}{i} = \frac{N_{\epsilon,p}}{i} \cdot \frac{N_{\epsilon,p}+1}{1} \cdot \cdots \cdot \frac{N_{\epsilon,p}+i-1}{i-1}. \tag{1}$$

Recall that $p^{h_{\epsilon+1}}$ is the largest power of p dividing $\epsilon+1$ and that $N_{\epsilon,p}$ is the smallest multiple of $p^{1+h_{\epsilon+1}}$ bigger than $\epsilon+1$. Hence, $k=N_{\epsilon,p}-\epsilon-1<p^{1+h_{\epsilon+1}}$ and the largest power of p dividing a given element of $\{1,\ldots,i\}$ is at most $p^{h_{\epsilon+1}}$. Now, for every $j\in\{1,\ldots,i-1\}$, the larger power of p dividing $N_{\epsilon,p}+j$ equals the largest power of p dividing j. The largest power of p dividing $N_{\epsilon,p}$ is equal to $p^{1+h_{\epsilon+1}}$, while the largest power of p dividing i is at most $p^{h_{\epsilon+1}}$. Hence, p divides $\binom{\epsilon+k+i}{i}$ by Eq. (1).

(2) Suppose $k=N-\epsilon-1$ where $\epsilon+1<N<N_{\epsilon,p}$. Let p^h be the largest power of p dividing N. Then $h\leqslant h_{\epsilon+1}$ and hence $N\geqslant\epsilon+1+p^h$, i.e. $p^h\leqslant N-\epsilon-1=k$. We have

$$\binom{\epsilon+k+p^h}{p^h}=\frac{N}{p^h}\cdot\frac{N+1}{1}\cdot\ldots\cdot\frac{N+p^h-1}{p^h-1}. \tag{2}$$

For every $j\in\{1,\ldots,p^h-1\}$, the largest power of p dividing $N+j$ equals the largest power of p dividing j. Recall that the largest power of p dividing N is equal to p^h. So, p is not a divisor of $\binom{\epsilon+k+p^h}{p^h}$ (recall (2)) and $p^h\in\{1,\ldots,k\}$. $\square$

Proposition 3.7.

(1) *If $char(\mathbb{K})=0$, then $\alpha^*_{2k}\notin W_{2k}$ for every $k\in\{1,\ldots,n\}$.*
(2) *Suppose $char(\mathbb{K})=p$. Then for fixed $\epsilon\in\mathbb{N}$ and variable $n>\epsilon$, $n^*=2(N_{\epsilon,p}-1)-\epsilon$ is the smallest value of n for which $n-\epsilon>0$ is even and $\alpha^*_{n-\epsilon}\in W_{n-\epsilon}$.*

Proof. (1) Suppose $char(\mathbb{K})=0$. If α^*_{2k} would belong to $W_{2k}=W_{2k,2k}$, then $\alpha^*_{2k}\in\ker(\theta_{2k,1})$ by Theorem 3.5. Now, $\theta_{2k,1}(\alpha^*_{2k})=(n-k+1)\cdot\alpha^*_{2k-2}\neq0$ since $n-k+1\neq0$.

(2) Suppose $char(\mathbb{K})=p$, $\epsilon\in\mathbb{N}$ and $2k:=n-\epsilon>0$ is even. By Theorem 3.5, $W_{2k}=W_{2k,2k}=\bigcap_{i=1}^k\ker(\theta_{k,i})$. Now, for every $i\in\{1,\ldots,k\}$, $\theta_{k,i}(\alpha^*_{2k})=\binom{n-k+i}{i}\cdot\alpha^*_{2k-2i}=\binom{\epsilon+k+i}{i}\cdot\alpha^*_{2k-2i}$. Claim (2) of the proposition now follows from Lemma 3.6. $\square$

Corollary 3.8. *Suppose $char(\mathbb{K})=p$, $\epsilon\in\mathbb{N}$ and $n=n^*=2(N_{\epsilon,p}-1)-\epsilon$. Then $\alpha^*_{n-\epsilon}\in R_{n-\epsilon}$.*

Proof. By Proposition 3.7, $\alpha^*_{n-\epsilon}\in W_{n-\epsilon}$.

Now, let $\bar e_1,\ldots,\bar e_n$ be n arbitrary linearly independent vectors of V such that $\langle\bar e_1,\ldots,\bar e_n\rangle$ is an n-dimensional totally isotropic subspace of V. Extend $(\bar e_1,\ldots,\bar e_n)$ to a hyperbolic basis $(\bar e_1,\bar f_1,\ldots,\bar e_n,\bar f_n)$ of V. Then $\alpha^*_{n-\epsilon}=\sum(\bar e_{i_1}\wedge\bar f_{i_1})\wedge\cdots\wedge(\bar e_{i_m}\wedge\bar f_{i_m})$, where the summation ranges over all $\binom{n}{m}$ subsets $\{i_1,\ldots,i_m\}$ of size $m:=\frac{n-\epsilon}{2}$ of $\{1,\ldots,n\}$. So, $\alpha^*_{n-\epsilon}\wedge\bar e_1\wedge\cdots\wedge\bar e_n=\sum(\bar e_{i_1}\wedge\bar f_{i_1})\wedge\cdots\wedge(\bar e_{i_m}\wedge\bar f_{i_m})\wedge\bar e_1\wedge\cdots\wedge\bar e_n=0$. $\square$

3.5. *An invariant subspace of $\bigwedge^{2k+1}V$, $k\in\{0,\ldots,n-1\}$*

Let $k\in\{0,\ldots,n-1\}$ and let $B=(\bar e_1,\bar f_1,\ldots,\bar e_n,\bar f_n)$ be a hyperbolic basis of V. For every $i\in\{1,\ldots,n\}$, let $\alpha_{2k,i}(B)$ be the vector

$$\sum(\bar e_{i_1}\wedge\bar f_{i_1})\wedge(\bar e_{i_2}\wedge\bar f_{i_2})\wedge\cdots\wedge(\bar e_{i_k}\wedge\bar f_{i_k}),$$

where the summation ranges over all $\binom{n-1}{k}$ subsets $\{i_1,i_2,\ldots,i_k\}$ of size k of $\{1,\ldots,n\}\setminus\{i\}$ satisfying $i_1<i_2<\cdots<i_k$. Let $R_{2k+1}(B)$ denote the $2n$-dimensional subspace of $\bigwedge^{2k+1}V$ generated by the $2n$ vectors $\bar e_i\wedge\alpha_{2k,i}(B)$, $\bar f_i\wedge\alpha_{2k,i}(B)$ $(i\in\{1,\ldots,n\})$. One readily verifies that if B_1 and B_2 are two hyperbolic bases of V such that $(B_1,B_2)\in\Omega_1\cup\cdots\cup\Omega_5$, then $R_{2k+1}(B_1)=R_{2k+1}(B_2)$. Hence, by Lemma 3.1, there exists a $2n$-dimensional subspace R^*_{2k+1} of $\bigwedge^{2k+1}V$ such that $R^*_{2k+1}=R_{2k+1}(B)$ for any hyperbolic basis B of V.

Lemma 3.9. *If $B = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$ and $B' = (\bar{e}'_1, \bar{f}'_1, \dots, \bar{e}'_n, \bar{f}'_n)$ are two hyperbolic bases of V such that $\bar{e}_1 = \bar{e}'_1$, then $\bar{e}_1 \wedge \alpha_{2k,1}(B) = \bar{e}_1 \wedge \alpha_{2k,1}(B')$.*

Proof. Let $B'' = (\bar{e}''_1, \bar{f}''_1, \bar{e}''_2, \bar{f}''_2, \dots, \bar{e}''_n, \bar{f}''_n)$ be the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}'_2 - f(\bar{e}'_2, \bar{f}_1) \cdot \bar{e}_1, \bar{f}'_2 - f(\bar{f}'_2, \bar{f}_1) \cdot \bar{e}_1, \dots, \bar{e}'_n - f(\bar{e}'_n, \bar{f}_1) \cdot \bar{e}_1, \bar{f}'_n - f(\bar{f}'_n, \bar{f}_1) \cdot \bar{e}_1)$ of V. Then obviously, $\bar{e}_1 \wedge \alpha_{2k,1}(B'') = \bar{e}_1 \wedge \alpha_{2k,1}(B')$. The vectors $\bar{e}''_2, \bar{f}''_2, \dots, \bar{e}''_n, \bar{f}''_n$ are f-orthogonal with $\bar{e}_1, \bar{f}_1$ and hence belong to the subspace $V' := \langle \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n \rangle$. If f' is the nondegenerate alternating bilinear form of V' induced by f, then $(\bar{e}''_2, \bar{f}''_2, \dots, \bar{e}''_n, \bar{f}''_n)$ and $(\bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$ are two hyperbolic bases of V'. So, by Section 3.4, $\alpha_{2k,1}(B) = \alpha_{2k,1}(B'')$. Together with $\bar{e}_1 \wedge \alpha_{2k,1}(B'') = \bar{e}_1 \wedge \alpha_{2k,1}(B')$, this implies that $\bar{e}_1 \wedge \alpha_{2k,1}(B) = \bar{e}_1 \wedge \alpha_{2k,1}(B')$. □

For every nonzero vector $\bar{x}$ of V, let $\eta_k(\bar{x})$ denote the vector $\bar{x} \wedge \alpha_{2k,1}(B)$ of R^*_{2k+1}, where B is an arbitrary hyperbolic basis of V having $\bar{x}$ as first component. We put $\eta_k(0)$ equal to the zero vector of $\bigwedge^{2k+1} V$.

Lemma 3.10. *η_k is a linear isomorphism from V to R^*_{2k+1}.*

Proof. (1) Obviously, $\eta_k(\bar{x}) = 0$ if and only if $\bar{x} = 0$.

(2) We prove that $\eta_k(\lambda\bar{x}) = \lambda \cdot \eta_k(\bar{x})$ for every $\lambda \in \mathbb{K}$ and every $\bar{x} \in V$. Obviously, this holds if $\bar{x} = 0$ or $\lambda = 0$. So, suppose $\bar{x} \neq 0$ and $\lambda \neq 0$. Let $B = (\bar{x}, \bar{f}_1, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$ be an arbitrary hyperbolic basis of V having $\bar{x}$ as a first component. Then $B' = (\lambda\bar{x}, \frac{\bar{f}_1}{\lambda}, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$ is also a hyperbolic basis of V. Now, $\eta_k(\lambda\bar{x}) = \lambda\bar{x} \wedge \alpha_{2k,1}(B') = \lambda\bar{x} \wedge \alpha_{2k,1}(B) = \lambda \cdot \eta_k(\bar{x})$.

(3) We prove that $\eta_k(\bar{x}_1 + \bar{x}_2) = \eta_k(\bar{x}_1) + \eta_k(\bar{x}_2)$ for any two vectors $\bar{x}_1$ and $\bar{x}_2$ of V satisfying $\lambda := f(\bar{x}_1, \bar{x}_2) \neq 0$. Let $B = (\bar{x}_1, \frac{\bar{x}_2}{\lambda}, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$ be a hyperbolic basis having $\bar{x}_1$ and $\frac{\bar{x}_2}{\lambda}$ as first two components. Then $B' = (\bar{x}_1 + \bar{x}_2, \frac{\bar{x}_2}{\lambda}, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$ and $B'' = (\bar{x}_2, -\frac{\bar{x}_1}{\lambda}, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$ are also hyperbolic bases of V. We have $\eta_k(\bar{x}_1 + \bar{x}_2) = (\bar{x}_1 + \bar{x}_2) \wedge \alpha_{2k,1}(B') = \bar{x}_1 \wedge \alpha_{2k,1}(B') + \bar{x}_2 \wedge \alpha_{2k,1}(B') = \bar{x}_1 \wedge \alpha_{2k,1}(B) + \bar{x}_2 \wedge \alpha_{2k,1}(B'') = \eta_k(\bar{x}_1) + \eta_k(\bar{x}_2)$.

(4) We prove that $\eta_k(\bar{x}_1 + \bar{x}_2) = \eta_k(\bar{x}_1) + \eta_k(\bar{x}_2)$ for any two linearly independent vectors $\bar{x}_1$ and $\bar{x}_2$ of V satisfying $f(\bar{x}_1, \bar{x}_2) = 0$. Let $B = (\bar{x}_1, \bar{f}_1, \bar{x}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3, \dots, \bar{e}_n, \bar{f}_n)$ be a hyperbolic basis of V having $\bar{x}_1$ as first component and $\bar{x}_2$ as third component. Then $B' = (\bar{x}_2, \bar{f}_2, \bar{x}_1, \bar{f}_1, \bar{e}_3, \bar{f}_3, \dots, \bar{e}_n, \bar{f}_n)$ and $B'' = (\bar{x}_1 + \bar{x}_2, \bar{f}_1, \bar{x}_2, -\bar{f}_1 + \bar{f}_2, \bar{e}_3, \bar{f}_3, \dots, \bar{e}_n, \bar{f}_n)$ are also hyperbolic bases of V. One can readily verify that $(\bar{x}_1 + \bar{x}_2) \wedge \alpha_{2k,1}(B'') = \bar{x}_1 \wedge \alpha_{2k,1}(B) + \bar{x}_2 \wedge \alpha_{2k,1}(B')$. Hence, $\eta_k(\bar{x}_1 + \bar{x}_2) = \eta_k(\bar{x}_1) + \eta_k(\bar{x}_2)$. □

Lemma 3.11. *The map $\theta \mapsto \tilde{\theta}_{2k+1}$ induces a representation of G on the vector space R^*_{2k+1} which is isomorphic to the natural representation of $G \cong Sp(2n, \mathbb{K})$ on V.*

Proof. In view of Lemma 3.10, it suffices to show that $\eta_k(\theta(\bar{x})) = \tilde{\theta}_{2k+1}(\eta_k(\bar{x}))$ for every $\theta \in G$ and every $\bar{x} \in V$. Obviously, this holds if $\bar{x} = 0$. So, suppose $\bar{x} \neq 0$ and consider a hyperbolic basis $B = (\bar{x}, \bar{f}_1, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$ of V having $\bar{x}$ as first component. Then $\tilde{\theta}_{2k+1}(\eta_k(\bar{x})) = \tilde{\theta}_{2k+1}(\bar{x} \wedge \sum \bar{e}_{i_1} \wedge \bar{f}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k} \wedge \bar{f}_{i_k}) = \theta(\bar{x}) \wedge \sum \theta(\bar{e}_{i_1}) \wedge \theta(\bar{f}_{i_1}) \wedge \cdots \wedge \theta(\bar{e}_{i_k}) \wedge \theta(\bar{f}_{i_k})$, and this is equal to $\eta_k(\theta(\bar{x}))$ since $(\theta(\bar{x}), \theta(\bar{f}_1), \theta(\bar{e}_2), \theta(\bar{f}_2), \dots, \theta(\bar{e}_n), \theta(\bar{f}_n))$ is a hyperbolic basis of V. □

3.6. Preparation of an inductive approach

Throughout this subsection we suppose that $n \geqslant k \geqslant 2$ and that $\bar{e}_1$ and $\bar{f}_1$ are two vectors of V such that $f(\bar{e}_1, \bar{f}_1) = 1$. Let V' denote the set of vectors of V which are f-orthogonal with $\bar{e}_1$ and $\bar{f}_1$ and let f' denote the nondegenerate alternating bilinear form of V' induced by f'. Let W'_{k-1} denote the subspace of $\bigwedge^{k-1} V'$ generated by all vectors of the form $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_{k-1}$ where $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1} \rangle$ is a $(k-1)$-dimensional subspace of V' which is totally isotropic with respect to f'. Let R'_{k-1} denote the set of all $\alpha \in W'_{k-1}$ with the property that $\alpha \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_{n-1} = 0$ for all $n-1$ vectors $\bar{v}_1, \dots, \bar{v}_{n-1}$ of V' such that $\langle \bar{v}_1, \dots, \bar{v}_{n-1} \rangle$ is an $(n-1)$-dimensional subspace of V' which is totally isotropic with respect to f'. Let G' be the group of isometries of (V', f') and let $\widetilde{G}'_{k-1}$ denote the subgroup of $GL(W'_{k-1})$ corresponding to G' (see Section 1). For every vector α of $\bigwedge^{k-1} V'$, let $\mu_k(\alpha)$ be the vector $\bar{e}_1 \wedge \alpha$ of $\bigwedge^k V$. Then μ_k defines an isomorphism between W'_{k-1} and the subspace $\mu_k(W'_{k-1})$ of W_k.

In some of the proofs of this subsection, we will make use of the following obvious facts which hold for all vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$, for all $i \in \{1, \ldots, k\}$ and all $\lambda_j \in \mathbb{K}$ with $j \in \{1, \ldots, k\} \setminus \{i\}$:

$$\bar{v}_1 \wedge \cdots \wedge \bar{v}_i \wedge \cdots \wedge \bar{v}_k = \bar{v}_1 \wedge \cdots \wedge \bar{v}_{i-1} \wedge \left(\bar{v}_i + \sum_{j \neq i} \lambda_j \bar{v}_j\right) \wedge \bar{v}_{i+1} \wedge \cdots \wedge \bar{v}_k, \tag{3}$$

$$\bar{v}_1 \wedge \cdots \wedge \bar{v}_i \wedge \bar{v}_{i+1} \wedge \cdots \wedge \bar{v}_k = \bar{v}_1 \wedge \cdots \wedge \bar{v}_{i+1} \wedge (-\bar{v}_i) \wedge \cdots \wedge \bar{v}_k \quad (i \neq k). \tag{4}$$

Now, every vector χ of $\bigwedge^k V$ can be written in a unique way as

$$\bar{e}_1 \wedge \bar{f}_1 \wedge \alpha(\chi) + \bar{e}_1 \wedge \beta(\chi) + \bar{f}_1 \wedge \gamma(\chi) + \delta(\chi),$$

where $\alpha(\chi) \in \bigwedge^{k-2} V'$, $\beta(\chi), \gamma(\chi) \in \bigwedge^{k-1} V'$ and $\delta(\chi) \in \bigwedge^k V'$. Notice that the maps $\alpha : \bigwedge^k V \to \bigwedge^{k-2} V'$, $\beta : \bigwedge^k V \to \bigwedge^{k-1} V'$, $\gamma : \bigwedge^k V \to \bigwedge^{k-1} V'$ and $\delta : \bigwedge^k V \to \bigwedge^k V'$ are linear.

Lemma 3.12. *If $\chi \in W_k$, then $\beta(\chi)$ and $\gamma(\chi)$ belong to W'_{k-1}.*

Proof. We will prove that $\beta(\chi) \in W'_{k-1}$. In a completely similar way one can also prove that $\gamma(\chi) \in W'_{k-1}$. By the linearity of β, we may restrict ourselves to the case where $\chi = \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ for some vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$ such that $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ is a k-dimensional subspace of V which is totally isotropic with respect to f. By Eqs. (3) and (4), we may suppose that $\bar{v}_2, \ldots, \bar{v}_k \in \langle \bar{f}_1, V' \rangle$ and $\bar{v}_3, \ldots, \bar{v}_k \in V'$. Let $\lambda_1, \lambda_2 \in \mathbb{K}$ such that $\bar{v}_1 - \lambda_1 \bar{e}_1 \in \langle \bar{f}_1, V' \rangle$ and $\bar{v}_2 - \lambda_2 \bar{f}_1 \in V'$. Then $\beta(\chi) = \lambda_1(\bar{v}_2 - \lambda_2 \bar{f}_1) \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_k \in W'_{k-1}$ since $\langle \bar{v}_2 - \lambda_2 \bar{f}_1, \bar{v}_3, \ldots, \bar{v}_k \rangle$ is a $(k-1)$- or $(k-2)$-dimensional subspace of V' which is totally isotropic with respect to f'. $\square$

Lemma 3.13. *Suppose U is a subspace of W_k which is stabilized by $\widetilde{G}_k$. Then $\mu_k^{-1}(U \cap \mu_k(W'_{k-1}))$ is a subspace of W'_{k-1} which is stabilized by $\widetilde{G}'_{k-1}$.*

Proof. Let α be an arbitrary vector of $\mu_k^{-1}(U \cap \mu_k(W'_{k-1}))$ and let $\tilde{\theta}$ be an arbitrary element of $\widetilde{G}'_{k-1}$ corresponding to an element $\theta \in G'$. We need to show that $\tilde{\theta}(\alpha) \in \mu_k^{-1}(U \cap \mu_k(W'_{k-1}))$.

We extend θ to an element $\bar{\theta}$ of G by defining $\bar{\theta}(\bar{e}_1) = \bar{e}_1$ and $\bar{\theta}(\bar{f}_1) = \bar{f}_1$. Let $\tilde{\bar{\theta}}$ be the element of $\widetilde{G}_k$ corresponding to $\bar{\theta}$. Then for every vector α' of W'_{k-1}, $\mu_k \circ \tilde{\theta}(\alpha') = \tilde{\bar{\theta}} \circ \mu_k(\alpha')$. Hence, $\tilde{\bar{\theta}}$ stabilizes $\mu_k(W'_{k-1})$.

Now, since $\mu_k(\alpha) \in U \cap \mu_k(W'_{k-1})$, also $\tilde{\bar{\theta}} \circ \mu_k(\alpha) \in U \cap \mu_k(W'_{k-1})$. Hence, $\tilde{\theta}(\alpha) = \mu_k^{-1} \circ \tilde{\bar{\theta}} \circ \mu_k(\alpha) \in \mu_k^{-1}(U \cap \mu_k(W'_{k-1}))$. $\square$

Lemma 3.14. $\mu_k(R'_{k-1}) = R_k \cap \mu_k(W'_{k-1})$.

Proof. (1) We prove that $R_k \cap \mu_k(W'_{k-1}) \subseteq \mu_k(R'_{k-1})$. Let α be an arbitrary element of W'_{k-1} such that $\bar{e}_1 \wedge \alpha \in R_k$. We need to show that $\alpha \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n = 0$ for all vectors $\bar{v}_2, \ldots, \bar{v}_n$ of V' such that $\langle \bar{v}_2, \ldots, \bar{v}_n \rangle$ is an $(n-1)$-dimensional subspace of V' which is totally isotropic with respect to f'. Since $\bar{e}_1 \wedge \alpha \in R_k$, we have $\bar{e}_1 \wedge \alpha \wedge \bar{f}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n = 0$ since $\langle \bar{f}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ is an n-dimensional subspace of V which is totally isotropic with respect to f. This implies that $\alpha \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n = 0$ as we needed to show.

(2) We prove that $\mu_k(R'_{k-1}) \subseteq R_k \cap \mu_k(W'_{k-1})$. Let α be an arbitrary element of R'_{k-1}. We need to show that $\bar{e}_1 \wedge \alpha \in R_k$, or equivalently, that $\bar{e}_1 \wedge \alpha \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n = 0$ for all vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ of V such that $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ is an n-dimensional subspace of V which is totally isotropic with respect to f. By Eqs. (3) and (4), we may suppose that $\bar{v}_2, \ldots, \bar{v}_n \in \langle \bar{e}_1, V' \rangle$ and $\bar{v}_3, \ldots, \bar{v}_n \in V'$. Let $\lambda \in \mathbb{K}$ such that $\bar{v}_2 - \lambda \bar{e}_1 \in V'$. Then $\langle \bar{v}_2 - \lambda \bar{e}_1, \bar{v}_3, \ldots, \bar{v}_n \rangle$ is an $(n-1)$- or $(n-2)$-dimensional subspace of V' which is totally isotropic with respect to f'. Now, $\bar{e}_1 \wedge \alpha \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n = \bar{e}_1 \wedge \alpha \wedge \bar{v}_1 \wedge (\bar{v}_2 - \lambda \bar{e}_1) \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n = 0$ since $\alpha \in R'_{k-1}$. $\square$

3.7. *Proof of Theorem 1.1*

The following proposition is precisely Theorem 1.1.

Proposition 3.15. *If U is a proper subspace of W_k which is stabilized by $\widetilde{G}_k$, then $U \subseteq R_k$.*

Proof. We will prove the proposition by induction on k. If $k = 1$, then $W_k = W_1 = \bigwedge^1 V = V$ and $U = 0$ since G acts transitively on the set of 1-spaces of V. Suppose therefore that $k \geqslant 2$ and that the proposition holds for smaller values of k.

Suppose by way of contradiction that there exists a $\chi_0 \in U \setminus R_k$. Then there exists an n-dimensional subspace $\langle \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n \rangle$ of V which is totally isotropic with respect to f such that $\chi_0 \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_n \neq 0$. Now, extend $(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)$ to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Let V', W'_{k-1}, R'_{k-1} and μ_k be as in Section 3.6.

We prove that $\mu_k^{-1}(U \cap \mu_k(W'_{k-1}))$ is a proper subspace of W'_{k-1}. If this would not be the case, then $\mu_k(W'_{k-1}) \subseteq U$ and hence U contains a vector of the form $\bar{e}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ where $\langle \bar{e}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ is a k-dimensional subspace of V which is totally isotropic with respect to f. Since G acts transitively on the set of all k-dimensional subspaces of V which are totally isotropic with respect to f, U must contain all vectors of the form $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ where $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ is a k-dimensional subspace of V which is totally isotropic with respect to V. This however would imply that $U = W_k$, which is impossible.

By Lemma 3.13 and the induction hypothesis, we now have:

$$\mu_k^{-1}\big(U \cap \mu_k\big(W'_{k-1}\big)\big) \subseteq R'_{k-1}. \tag{5}$$

Now, we can write χ_0 in a unique way as

$$\chi_0 = \bar{e}_1 \wedge \bar{f}_1 \wedge \alpha(\chi_0) + \bar{e}_1 \wedge \beta(\chi_0) + \bar{f}_1 \wedge \gamma(\chi_0) + \delta(\chi_0),$$

where $\alpha(\chi_0) \in \bigwedge^{k-2} V'$, $\beta(\chi_0), \gamma(\chi_0) \in \bigwedge^{k-1} V'$ and $\delta(\chi_0) \in \bigwedge^k V'$. We prove that $\chi_1 := \bar{e}_1 \wedge \bar{f}_1 \wedge \alpha(\chi_0) + \delta(\chi_0)$ belongs to U and satisfies $\chi_1 \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_n \neq 0$. In order to achieve this goal, it suffices to prove that $\bar{e}_1 \wedge \beta(\chi_0) \in U$, $\bar{f}_1 \wedge \gamma(\chi_0) \in U$ and $\beta(\chi_0) \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_n = 0$.

Let θ be the unique element of G mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V to the hyperbolic basis $(\bar{e}_1 + \bar{f}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then $\tilde{\theta}_k(\chi_0) = \chi_0 + \bar{f}_1 \wedge \beta(\chi_0)$. Since $\chi_0 \in U$, also $\tilde{\theta}_k(\chi_0) \in U$ and hence $\bar{f}_1 \wedge \beta(\chi_0) \in U$. Let θ' be the element of G mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V to the hyperbolic basis $(-\bar{f}_1, \bar{e}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then since $\bar{f}_1 \wedge \beta(\chi_0) \in U$, also $\bar{e}_1 \wedge \beta(\chi_0) = \tilde{\theta}'_k(\bar{f}_1 \wedge \beta(\chi_0)) \in U$. In a similar way one proves that $\bar{f}_1 \wedge \gamma(\chi_0) \in U$. Now by Lemma 3.12, $\beta(\chi_0) \in W'_{k-1}$. Hence, $\bar{e}_1 \wedge \beta(\chi_0) \in U \cap \mu_k(W'_{k-1})$. By (5), $\beta(\chi_0) \in R'_{k-1}$. As a consequence, $\beta(\chi_0) \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_n = 0$.

We now inductively define vectors $\chi_i \in \bigwedge^k V$ for every $i \in \{2, \ldots, n\}$. Let V'_i be the subspace of V which is f-orthogonal with $\langle \bar{e}_i, \bar{f}_i \rangle$. We can write χ_{i-1} in a unique way as

$$\chi_{i-1} = \bar{e}_i \wedge \bar{f}_i \wedge \alpha(\chi_{i-1}) + \bar{e}_i \wedge \beta(\chi_{i-1}) + \bar{f}_i \wedge \gamma(\chi_{i-1}) + \delta(\chi_{i-1}),$$

where $\alpha(\chi_{i-1}) \in \bigwedge^{k-2} V_i'$, $\beta(\chi_{i-1}), \gamma(\chi_{i-1}) \in \bigwedge^{k-1} V_i'$ and $\delta(\chi_{i-1}) \in \bigwedge^k V_i'$. We define

$$\chi_i := \bar{e}_i \wedge \bar{f}_i \wedge \alpha(\chi_{i-1}) + \delta(\chi_{i-1}).$$

With a completely similar reasoning as above, we can (inductively) prove that $\chi_i \in U$ and $\chi_i \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_n \neq 0$ for every $i \in \{2, \ldots, n\}$. If k is odd, then $\chi_n = 0$. If k is even, then χ_n is a sum of terms of the form $\lambda \cdot (\bar{e}_{i_1} \wedge \bar{f}_{i_1}) \wedge \cdots \wedge (\bar{e}_{i_l} \wedge \bar{f}_{i_l})$ where $l = \frac{k}{2}$. In any case, we have $\chi_n \wedge \bar{f}_1 \wedge \cdots \wedge \bar{f}_n = 0$, which is a contradiction.

Hence, $U \subseteq R_k$. $\square$

3.8. *Proof of Theorem 1.5*

(I) Let $\epsilon \in \mathbb{N}$ be fixed and $n > \epsilon$ be variable. We suppose that there exists a value of $n > \epsilon$ for which $R_{n-\epsilon} \neq 0$. We moreover suppose that n is the smallest value for which this is the case. Then $k := n - \epsilon \geqslant 2$. Let $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ be a hyperbolic basis of V and let V', W_{k-1}', R_{k-1}' and μ_k be as in Section 3.6. By the minimality of n, we have $R_{k-1}' = 0$.

Let χ be an arbitrary vector of R_k. Then $\chi = \bar{e}_1 \wedge \bar{f}_1 \wedge \alpha(\chi) + \bar{e}_1 \wedge \beta(\chi) + \bar{f}_1 \wedge \gamma(\chi) + \delta(\chi)$ where $\alpha(\chi) \in \bigwedge^{k-2} V'$, $\beta(\chi), \gamma(\chi) \in \bigwedge^{k-1} V'$ and $\gamma(\chi) \in \bigwedge^k V'$. Consider the element θ of G mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ of V to the hyperbolic basis $(\bar{e}_1, \bar{f}_1 + \bar{e}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then $\tilde{\theta}_k(\chi) = \chi + \bar{e}_1 \wedge \gamma(\chi)$. Since $\chi \in R_k$, also $\tilde{\theta}_k(\chi) \in R_k$ and hence $\bar{e}_1 \wedge \gamma(\chi) \in R_k$. By Lemma 3.12, $\bar{e}_1 \wedge \gamma(\chi) \in \mu_k(W_{k-1}')$. Hence, by Lemma 3.14, $\gamma(\chi) \in R_{k-1}'$, i.e. $\gamma(\chi) = 0$. In a completely similar way one can prove that $\beta(\chi) = 0$.

What we have just done, we can also do for any pair $(\bar{e}_i, \bar{f}_i)$, $i \in \{1, \ldots, n\}$. We can conclude:

$(*)$ For every $i \in \{1, \ldots, n\}$ and every $\chi \in R_k$, χ can be written in the form $\bar{e}_i \wedge \bar{f}_i \wedge \alpha_i(\chi) + \delta_i(\chi)$ where $\alpha_i(\chi) \in \bigwedge^{k-2}\langle \bar{e}_1, \bar{f}_1, \ldots, \hat{\bar{e}}_i, \hat{\bar{f}}_i, \ldots, \bar{e}_n, \bar{f}_n \rangle$ and $\delta_i(\chi) \in \bigwedge^k \langle \bar{e}_1, \bar{f}_1, \ldots, \hat{\bar{e}}_i, \hat{\bar{f}}_i, \ldots, \bar{e}_n, \bar{f}_n \rangle$.

If k is odd, then $(*)$ implies that $R_k = 0$, a contradiction. Hence, $k = 2m$ is even. By $(*)$, every element χ of R_k is of the form $\sum \lambda_I \cdot \bar{e}_{i_1} \wedge \bar{f}_{i_1} \wedge \cdots \wedge \bar{e}_{i_m} \wedge \bar{f}_{i_m}$, with the summation ranging over all subsets $I = \{i_1, \ldots, i_m\}$ of size m of $\{1, \ldots, n\}$ satisfying $i_1 < i_2 < \cdots < i_m$. We will now show that all the coefficients λ_I are equal to each other.

Suppose first that I_1 and I_2 are two subsets of size m of $\{1, 2, \ldots, n\}$ such that $|I_1 \cap I_2| = m - 1$. Without loss of generality, we may suppose that $I_1 \setminus I_2 = \{1\}$ and $I_2 \setminus I_1 = \{2\}$. Write $\sum \lambda_I \cdot \bar{e}_{i_1} \wedge \bar{f}_{i_1} \wedge \cdots \wedge \bar{e}_{i_m} \wedge \bar{f}_{i_m}$ in the form

$$\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 \wedge \alpha + \bar{e}_1 \wedge \bar{f}_1 \wedge \beta + \bar{e}_2 \wedge \bar{f}_2 \wedge \gamma + \delta,$$

where $\alpha \in \bigwedge^{k-4}\langle \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n \rangle$, $\beta, \gamma \in \bigwedge^{k-2}\langle \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n \rangle$ and $\delta \in \bigwedge^k \langle \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n \rangle$. [If $k = 2$, then we omit the term $\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 \wedge \alpha$.] Let θ denote the element of G mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ of V to the hyperbolic basis $(\bar{e}_1 + \bar{e}_2, \bar{f}_1, \bar{e}_2, -\bar{f}_1 + \bar{f}_2, \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then $\tilde{\theta}_k(\chi) = \chi + \bar{e}_2 \wedge \bar{f}_1 \wedge (\beta - \gamma)$. Since $\chi \in R_k$, also $\tilde{\theta}_k(\chi) \in R_k$ and hence $\bar{e}_2 \wedge \bar{f}_1 \wedge (\beta - \gamma) \in R_k$. By $(*)$, $\beta = \gamma$. Hence, $\lambda_{I_1} = \lambda_{I_2}$.

Consider now the most general case and let I_1 and I_2 be two arbitrary subsets of size m of $\{1, \ldots, n\}$. Put $|I_1 \cap I_2| = m - l$. Then there exist $l + 1$ subsets $J_0, \ldots, J_l$ of size m of $\{1, \ldots, n\}$ such that $J_0 = I_1$, $J_l = I_2$ and $|J_{i-1} \cap J_i| = m - 1$ for every $i \in \{1, \ldots, l\}$. By the previous paragraph, we know that $\lambda_{I_1} = \lambda_{J_0} = \lambda_{J_1} = \cdots = \lambda_{J_l} = \lambda_{I_2}$.

Hence, R_k is 1-dimensional and equal to $\langle \alpha_{2m}^* \rangle$. By Proposition 3.7 and Corollary 3.8, we then know that $char(\mathbb{K}) = p$ and $n = n^* := 2(N_{\epsilon,p} - 1) - \epsilon$. So, we have proved the following proposition:

Proposition 3.16.

(1) *If $char(\mathbb{K}) = 0$, then $R_k = 0$ for every $k \in \{1, \ldots, n\}$.*

(2) *Suppose char*($\mathbb{K}$) $= p$. *Then for fixed* ϵ *and variable* $n > \epsilon$, $R_{n-\epsilon} = 0$ *if* $\epsilon < n < n^*$. *If* $n = n^*$, *then* $R_{n-\epsilon} = \langle \alpha^*_{n-\epsilon} \rangle$.

(II) Let $\epsilon \in \mathbb{N}$ be fixed and $n > \epsilon$ be variable. Let $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ be a hyperbolic basis of V and let R'_{k-1} and μ_k be as defined in Section 3.6. By Lemma 3.14, $\bar{e}_1 \wedge R'_{k-1} \subseteq R_k$ and $\bar{f}_1 \wedge R'_{k-1} \subseteq R_k$. This fact in combination with Proposition 3.16(2) gives us the following:

Proposition 3.17. *Suppose char*($\mathbb{K}$) $= p$. *Let* $\epsilon \in \mathbb{N}$ *be fixed and* $n > \epsilon$ *be variable. Then for all* $n > \epsilon$, $\dim(R_{n+1-\epsilon}) \geqslant 2 \cdot \dim(R_{n-\epsilon})$. *As a consequence,* $R_{n-\epsilon} = 0$ *if and only if* $\epsilon < n < n^* := 2(N_{\epsilon,p} - 1) - \epsilon$.

(III) Suppose *char*($\mathbb{K}$) $= p$ and $\epsilon \in \mathbb{N}$. As before, put $n^* = 2(N_{\epsilon,p} - 1) - \epsilon$. Put $n = n^* + 1$ and $2k + 1 = n - \epsilon$.

Proposition 3.18. *We have* $R_{2k+1} = R^*_{2k+1}$.

Proof. Let $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ be a hyperbolic basis of V. By Lemma 3.14 and the explicit description of the largest proper submodule in the case $n = n^*$ (see Proposition 3.16), we know that $\bar{e}_i \wedge \alpha_{2k,i}(B) \in R_{2k+1}$ and $\bar{f}_i \wedge \alpha_{2k,i}(B) \in R_{2k+1}$ for every $i \in \{1, \ldots, n\}$. Hence, $R^*_{2k+1} \subseteq R_{2k+1}$. We will now also prove that $R_{2k+1} \subseteq R^*_{2k+1}$.

Let χ be an arbitrary vector of R_{2k+1}. For every $i \in \{1, \ldots, n\}$, let $\alpha_i \in \bigwedge^{2k-1} \langle \bar{e}_1, \bar{f}_1, \ldots, \hat{\bar{e}}_i, \hat{\bar{f}}_i, \ldots, \bar{e}_n, \bar{f}_n \rangle$, $\beta_i, \gamma_i \in \bigwedge^{2k} \langle \bar{e}_1, \bar{f}_1, \ldots, \hat{\bar{e}}_i, \hat{\bar{f}}_i, \ldots, \bar{e}_n, \bar{f}_n \rangle$ and $\delta_i \in \bigwedge^{2k+1} \langle \bar{e}_1, \bar{f}_1, \ldots, \hat{\bar{e}}_i, \hat{\bar{f}}_i, \ldots, \bar{e}_n, \bar{f}_n \rangle$ such that $\chi = \bar{e}_i \wedge \bar{f}_i \wedge \alpha_i + \bar{e}_i \wedge \beta_i + \bar{f}_i \wedge \gamma_i + \delta_i$. We will now prove that β_i and γ_i are multiples of $\alpha_{2k,i}(B)$. Since $2k + 1$ is odd and $\chi \in \bigwedge^{2k+1} V$, this fact then implies that $\chi = \sum_{i=1}^{n} (\bar{e}_i \wedge \beta_i + \bar{f}_i \wedge \gamma_i) \in R^*_{2k+1}$.

Consider the element $\theta \in G$ mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V to the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_i + \bar{f}_i, \bar{f}_i, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Then $\tilde{\theta}_k(\chi) = \chi + \bar{f}_i \wedge \beta_i$. Since $\chi \in R_{2k+1}$, also $\tilde{\theta}_k(\chi) \in R_{2k+1}$ and hence $\bar{f}_i \wedge \beta_i \in R_{2k+1}$. In a similar way one proves that $\bar{e}_i \wedge \gamma_i \in R_{2k+1}$. By Lemmas 3.12 and 3.14 and the explicit description of the largest proper submodule in the case $n = n^*$, we then know that β_i and γ_i are multiples of $\alpha_{2k,i}(B)$. As mentioned before this fact implies that $\chi = \sum_{i=1}^{n} (\bar{e}_i \wedge \beta_i + \bar{f}_i \wedge \gamma_i) \in R^*_{2k+1}$. Hence, also $R_{2k+1} \subseteq R^*_{2k+1}$. $\square$

Lemma 3.11 and Proposition 3.18 then imply the following.

Proposition 3.19. *Suppose char*($\mathbb{K}$) $= p$, $\epsilon \in \mathbb{N}$, $n^* = 2(N_{\epsilon,p} - 1) - \epsilon$ *and* $n = n^* + 1$. *Then the representation* $\mathcal{R}^{(2)}_{n-\epsilon}$ *is isomorphic to the natural representation of* $G \cong Sp(2n, \mathbb{K})$ *on* V.

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